On Determination of a Regional Vertical Datum by Combination of EGM, Local Gravity and GPS/leveling Data

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Key words: geodetic heights, orthometric heights, normal heights, geoid, quasi-geoid, gravity field, reference ellipsoid

SUMMARY

Global Navigation Satellite Systems (GNSS) allow for precise 3D positioning with respect to a selected geocentric reference ellipsoid. Still, physical heights referring to a regional vertical datum are widely used by surveying and mapping agencies due to their link to the Earth's gravity field that must be taken into the account for hydrological and other projects. The transformation of geodetic heights measured by GNSS receivers into national height systems became an everyday task for surveyors worldwide. The determination of a local or regional vertical datum required for the height transformation is discussed in this contribution. Determination methods based on combination of global geopotential models, local gravity and GPS/leveling data are described including an impact of recent advances in global gravity field modeling and increasing availability of GPS/leveling data. Both concepts currently used in geodesy (orthometric heights/geoid and normal heights/quasi-geoid) are reviewed.

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1. INTRODUCTION

Physically-meaningful heights referred to a certain sea level are used worldwide in geodesy, surveying engineering, mapping, land cadastre and geographic information systems – just to name some of many application areas. Above mean sea level (AMSL) heights survived positioning using Global Navigation Satellite Systems (GNSS) that have become a prominent positioning technique in recent years. GNSS data can be used for 3-D positioning with a height referring to an adopted Earth's reference ellipsoid. Such geodetic (ellipsoidal) heights can be determined with a relatively high precision but they must be converted into AMSL heights in a height system used in a specific country or region. This manuscript reviews data and methodologies used in geodesy for determination of parameters that transform geodetic heights into orthometric or normal heights in a particular regional height system. A project of the World Height System (WHS) is also briefly mentioned in the manuscript. The local quasi-geoid model in the Czech Republic is used as an example of the regional vertical datum estimated by combining ground gravity, Earth gravitational model and GNSS/leveling data.

2. OVERVIEW OF HEIGHTS USED IN GEODESY AND SURVEYING

To define the 3-D position of points and objects on or above the Earth's surface, different coordinate systems (local/global, natural/model) and coordinate types (Cartesian/curvilinear) are used. In many applications, vertical separations (heights) from an equipotential surface of the Earth's gravity field represented by the mean sea level (geoid) or from model geometry (biaxial geocentric reference ellipsoid) are used. In the former case, we recognize *orthometric heights* determined traditionally by spirit leveling, in the latter case we refer to *geodetic heights* estimated nowadays by GNSS. Orthometric heights are reckoned along a curved local plumb line of the Earth's gravity field, geodetic heights define a separation of a point above the reference ellipsoid along its straight surface normal. Both heights can be related through the separation of the reference ellipsoid and the geoid called a *geoidal height* (undulation).

Orthometric heights are based on leveling; however, corrections must be applied to leveled height differences. The corrections are estimated from gravity values observed on the Earth's surface and continued inside topographic masses. Since the computation relies on the mass density of the topographic masses, the orthometric heights cannot rigorously be determined. Depending on the approximation used for their evaluation, orthometric heights get different adjectives such as Helmert or normal orthometric heights. Despite some problems with their estimation, the orthometric heights are still used by many countries, e.g., in Canada and USA. Recent years saw some significant advances in the theory of orthometric heights and large improvements in availability of data required for their estimation, e.g., (Tenzer et al. 2005).

In the period after the World War II, important advances in the theory of AMSL heights had been made. To avoid known problems with the estimation of precise orthometric heights (and the geoid), a new theory was introduced by the Russian geodesist Molodensky in 1960. In his theory, orthometric heights were replaced by *normal heights* and the geoid was replaced by the quasi-geoid. The important advantage of this new theory is that no a priori assumptions concerning the topographic mass density must be made. Moreover, no ground gravity data must be collected since the normal heights rely on a model (normal) gravity field generated by the homogeneous reference ellipsoid of the same mass and rotation as the actual Earth. The main advantage of the normal heights is their determinability; their disadvantage is a little bit looser relation to the physical properties of the Earth since they refer to the so-called quasi-geoid (or quasi mean sea level). In contrary to orthometric heights, they are reckoned along plumb lines of the normal gravity field that are in practice approximated by ellipsoidal surface normals. Geodetic heights can be then related to normal heights through the separation of the quasi-geoid from the reference ellipsoid called in geodesy a *height anomaly*. The geoidal height and the height anomaly agree over land areas at the level of dm with extreme values in high mountains at the level of 1.5 m. The quasi-geoid coincides with the geoid over oceans.

Every day surveyors around the world solve the problem of transforming geodetic heights as obtained by GNSS into a regional height system of particular AMSL heights. Not only the type of heights is important, but also an offset of the regional height reference (specific tide gauge) from the adopted international reference must be known. In the following sections, fundamentals from the theory of estimating geoidal heights and height anomalies are given.

2.1 Geoidal heights

The theory for determining geoidal heights is quite well developed in geodesy. Basically, the geoidal undulation (required for transformation of the geodetic height into the orthometric height) can be determined from gravity observations collected traditionally at the Earth's surface, more recently by airborne sensors and most recently also by spaceborne techniques. Some other data types, such as sea surface heights from satellite altimetry and/or deflections of the vertical can be considered as well. Different data types have various properties such as the geographical coverage, accuracy and spectral content. For example, ground data are very accurate but their distribution is mainly local or regional. Vast areas of the Earth's surface are not covered at all (oceans, polar and tropical regions, high mountains). Airborne data are less accurate, but they filled recently some impassable areas of the world. Satellite data have the best geographical coverage (global but polar caps), but they are highly attenuated by the large distance from the Earth.

In practice, all data types are combined. Spaceborne data are usually combined with ground and airborne data that result a combined global Earth Gravitational Model (EGM). Its most recent version is EGM08 (Pavlis et al. 2008) that describes the Earth's gravitational potential by using a spherical harmonic series up to degree and order 2159 (equiangular resolution of 5 arcmin). This model can be used for evaluation of the low-frequency part of the (quasi-)geoid with the estimated accuracy of about ±15 cm for the given angular resolution of 5 arcmin. To get a higher accuracy and/or resolution, local gravity data can further be considered.

The high-frequency gravimetric geoid is computed from geographically-limited ground and/or airborne gravity data. The mathematical model for its determination from ground gravity is based on transformation of observed discrete values of gravity into the gravity potential related to the international reference ellipsoid. This transformation is usually performed in two steps: (1) observed values of ground gravity are downward continued to the ellipsoid, and (2) gravity at the ellipsoid is transformed into the corresponding potential. Each of these two steps represents a solution to one geodetic boundary-value problem (GBVP) of the potential theory, namely the first and second GBVP. Thus, two GBVPs must be formulated and solved that requires the numerical evaluation of two surface integrals. Alternatively, a mathematical model in the form of a single Fredholm integral equation of the first kind can be formulated transforming ground gravity disturbances into the disturbing gravity potential at the surface of the reference ellipsoid directly.

The problem at hand is to determine the disturbing gravity potential T at the reference ellipsoid from observations of gravity g at the topography. It is assumed that (1) the potential T is harmonic everywhere outside the reference ellipsoid, i.e., both T and its functionals can be represented at every point outside the reference ellipsoid by a convergent series of some harmonic base functions (such as spherical harmonics), and (2) harmonics of degree 0 and 1 are equal to zero. That can be achieved (1) through gravity reduction (direct topographic and atmospheric effects), and (2) by selecting a properly oriented and positioned Earth's reference ellipsoid. The problem of deriving a harmonic gravity field outside the reference ellipsoid is considered to be outside the scope of this article. It is acknowledged that this is a serious problem of geoid determination (mainly due to the unknown topographic mass density distribution) that eventually resulted in formulation of the Molodensky theory, see Section 2.2. The estimated value of the disturbing potential T, corrected for the indirect topographic and atmospheric effects, can easily be converted into the geoidal height using the well-known Bruns formula.

This contribution does not aim to present the problem of the geoid determination in all its complexity. Some approximations in physics and geometry are made for the formulation of the model, e.g., it is assumed that the value of the geopotential at the geoid is both known and constant in time. Other approximations involve formulations of the boundary-value problems including their solution integrals. These simplifications and approximations do not affect the idea of the solution and keep its formulation on a level that can easily be followed by a reader.

Since only geographically-limited ground data (reduced for reference gravity from EGM) are available for determination of the residual geoid, the reference ellipsoid can be approximated locally by a geocentric sphere (spherical approximation) with the radius adjusted to fit locally the reference ellipsoid. The spherical approximation is introduced to simplify the formulation of solution integrals. For the spherical approximation, geocentric spherical coordinates (r, ϕ, λ) are used for definition of a position of points of interest. In this coordinate system, the position of each point is uniquely described by its geocentric radius r, geocentric latitude ϕ and geocentric longitude λ.

In the spherical approximation, the following formulation can be given to the problem of the geoid determination from ground gravity data with known geodetic heights h

$$\iota\ {}^{2}T r(R,0\ ,\ ), \qquad (1.1)$$

$$\int g R, h r R h 2 = + \left. \frac{\partial T r(\ ,,\ )}{\partial r} \right|_{r} \qquad (1.2)$$

$$T r(Q) r;\quad ,=\#\circledR\left(\ ^{23}\right) \qquad (1.3)$$

where δg stands for the gravity disturbances that can be determined at the height $R+h$ from observed and reduced ground gravity data. The disturbing gravity potential T at the surface of the approximating geocentric sphere is unknown in this problem. In this case, both boundaries represented by the topography and the sphere of radius R are known and could be used for the solution. The spherical approximation of the reference ellipsoid is acceptable due to the local character of residual geoid computations.

The transformation of observed reduced values $\delta g(R+h,\phi,\lambda)$ into the sought function $T(R,\phi,\lambda)$ can be done in one step (Ψ is the integration domain – spherical cap)

$$\int g\left(R h T R K Y,',' \right) \sin \frac{1}{4M} d' \blacklozenge' \blacklozenge = \left(<1\ .\quad\right)\quad\left(\quad\right) \qquad \frac{R}{R h +} \qquad (1.4)$$

The corresponding integral kernel (Green function) K is of the spectral form (Novák 2003)

$$K\left(R Y, \Phi Y\right) \cos \sum_{n=2}^{\iota} \frac{n+1}{n+2}\ {}^{n+2}\ {}_{n}\left(\qquad\right) \qquad (1.5)$$

Since ground gravity data are available only over limited regions, the spherical cap integration is usually applied calculating truncation errors on the basis of an available global EGM in terms of the spherical harmonic expansion. Legendre polynomials P_n are functions of the spherical distance ψ between the computation and integration points. The geoidal height is finally given by the Bruns formula that reads approximately as follows:

$$N(\ ,\ .) = \frac{T R,,)}{B\left(A r_e,\ \right)} \qquad (1.6)$$

It should be remarked that another one-step spherical approach was derived for band-limited airborne gravity data by Novák and Heck (2002) but their approach is based on the direct integration while the approach for ground data in this contribution must unfortunately rely on the solution of the Fredholm integral formula of the first kind.

2.2 Height anomalies

To transform geodetic heights into normal heights, height anomalies must be known. For this purpose, local or regional models of the quasi-geoid are being estimated. As in the case of the geoid in Section 2.1, the quasi-geoid is also estimated *per partes* in terms of its spectrum. The low-frequency (long-wavelength) component is again based on EGM and the remaining high-frequency (short-wavelength) of the quasi-geoid is computed from available regional gravity data. The main difference from the estimation of the geoidal height is that ground gravity data are reduced neither for the direct topographic effect nor downward continued to the ellipsoid. The theory for computation of the height anomalies from ground gravity data was also developed by Molodensky. The entire apparatus needed for the task would require in all its complexity too much space in this contribution. Only basic formulas in the standard spherical approximation are reviewed in this section.

The GBVP is used for the estimation of the residual height anomaly. This apparatus requires no masses outside the Earth that can be achieved by additional reduction of ground gravity data for the direct atmospheric effect. Since the gravitational effect of the global atmospheric masses is included in EGM, only its residual component should further be taken into the account. Its magnitude is very small (tens of μGal) with a small effect on the quasi-geoid. The behavior of the residual disturbing gravity potential T everywhere outside the Earth r_t is then controlled by the Laplace differential equation (Heiskanen and Moritz 1967)

$$\nabla^2 T(r,\Omega) = 0 \ , \quad r > r_t \ . \tag{1.7}$$

The boundary condition to the homogeneous elliptical equation (1.7) for the solution of the unknown function T at the topography r_t represents the fundamental gravimetric equation that reads in the spherical approximation as follows, e.g., (Martinec and Vaníček 1996):

$$\delta g(r_t,\Omega) = -\left.\frac{\partial T(r,\Omega)}{\partial r}\right|_{r=r_t} \tag{1.8}$$

Residual gravity disturbances can be computed (Heiskanen and Moritz 1967)

$$\delta g(r_t,\Omega) = g(r_t,\Omega) - \gamma(r_t,\Omega) - \frac{GM}{R^2}\sum_{n=2}^{l}(n-1)\left(\frac{R}{r_t}\right)^{n+2} T_n(\Omega) \tag{1.9}$$

with observed ground gravity g reduced for the residual atmospheric effect, the geocentric gravitational constant GM and surface harmonics T_n of the disturbing potential derived from EGM08 and GRS80. The solution for the unknown function T exists and is unique when T is regular at infinity and when δg does not contain zero-degree and first-degree harmonics, e.g., (Heiskanen and Moritz 1967). This condition can be satisfied by the proper selection of the reference ellipsoid, thus normal gravity γ.

In contrast to the classical Stokes problem (geoid), the boundary values in Eq. (1.9) apply to the physical surface of the Earth with relatively complex geometry. The solution in terms of the boundary integral is then more complicated since the corresponding integral kernel cannot be constructed analytically. The solution can be expressed in the form of an infinite series with a zero-degree term corresponding to the spherical boundary (Molodensky et al. 1960)

$$T(r_t) = \frac{R}{4\pi} \iint_{\Phi} \Delta g(r_t') \, J(\psi) \sin\psi' \, d\psi' \, d\alpha' + \frac{R}{4\pi} \iint_{\Phi} \sum_{n=1}^{\infty} G_n(r_t') \, J(\psi) \sin\psi' \, d\psi' \, d\alpha' , \tag{1.10}$$

and the residual height anomaly can be again obtained through the Bruns formula with normal gravity referred to the telluroid rather than the ellipsoid, see Eq. (1.6).

Due to their small magnitude, G_n terms for $n>1$ are usually neglected. The G_1 term can be computed according to (Heiskanen and Moritz 1967) as follows:

$$G_1(r_t) = \frac{R^2}{2\pi} \iint_{\Psi} \frac{H(r_t') - H(r_t)}{L^3(r_t, r_t')} \, \Delta g(r_t') \sin\psi' \, d\psi' \, d\alpha' \tag{1.11}$$

with the Euclidean distance L between the computation point r_t and integration point r_t'. Due to the fast attenuation of the integral in Eq. (1.11), the spherical cap Ψ with radius of 1 arcdeg is usually used for its numerical evaluation. The kernel function J for computation of the residual potential in Eq. (1.10) is the spheroidal Hotine function defined through the infinite series of Legendre polynomials P_n

$$J(\psi) = \sum_{n=1}^{\infty} \frac{2n+1}{n+1} P_n(\cos\psi) \tag{1.12}$$

3. HEIGHT TRANSFORMATION AND GPS/LEVELING DATA

The conversion of geodetic heights h to orthometric heights H^O can be done easily

$$H^O(\varphi, \lambda) = h(\varphi, \lambda) - N(\varphi, \lambda) \tag{1.13}$$

and of geodetic heights h to normal heights H^N through

$$H^N(\varphi, \lambda) = h(\varphi, \lambda) - \zeta(\varphi, \lambda) \tag{1.14}$$

Measuring currently the geodetic heights h with the cm-level accuracy (GNSS positioning), the final accuracy of the orthometric or normal heights will depend namely on the accuracy of the geoidal height N or height anomaly ς.

Observing leveling points by GNSS, point values of geoidal heights or height anomalies can be considered as observables. These values can be then used for adjusting the local geoid or quasi-geoid model to the regional height system, namely for estimation of its offset. This offset is due to inconsistencies in the definition of the W_0 value (potential of the geoid once adopted through realization of the mean sea level and once through used EGM). Estimated values of N or ς can be used for modifying the computed gravimetric solution. The offset of the regional height system is always estimated and taken into the account; in some cases, a linear trend is also estimated. Higher-degree polynomials are then used for the determination of transformation (corrective) surfaces used merely for height transformation purposes.

Since the regional height systems rely on different realizations of the mean sea level (besides using different AMSL heights), offsets at their boundaries are often encountered. To change this situation, there is an attempt to create a homogeneous World Height System (WHS) based on one global vertical datum. This system has been investigated by geodesists within a project of the International Association of Geodesy (IAG). There are many problems related to the establishment of a worldwide height system such as estimating and adopting a conventional value of the gravity potential corresponding to the geoid (W_0), combination of available gravity and geometric data, determination of the global geoid and estimation of offsets for various regional height systems.

4. CASE STUDY: THE QUASI-GEOID MODEL OF THE CZECH REPUBLIC

This section describes an example of the regional height system used in the Czech Republic and its relationship to the local quasi-geoid model in the Czech Republic estimated from local ground gravity and elevation data, EGM08 as well as GNSS/leveling data.

4.1 Input data

Local ground gravity and elevation data used for evaluation of the residual quasi-geoid are given at the geographical grid with the equiangular resolution of 30 arcsec (approximately 1 km x 1 km at the Earth's surface). The database was compiled from existing national gravity and elevation data sets of the Czech Republic, Slovakia, Germany, Austria and Poland. Their accuracy, resolution and form were not the same over the entire data area. The accuracy of predicted mean values of gravity at the regular grid of geographical coordinates is characterized by the rms error at the level of ±1 mGal. This value applies namely to gravity data over the territory of former Czechoslovakia where observed data adjusted within the national gravimetric network were used for predicting surface area means over the grid.

The standard procedure consisting of gravity reduction, prediction and surface averaging was used in compilation of the database with height information defined through the local elevation model. The local elevation model consists of SRTM (Shuttle Radar Topography Mission) heights corrected and completed for heights from the national elevation models. This approach, that is inevitable for application of the computational model based on integral equations introduced in Section 2, leads to the loss of estimated stochastic properties of adjusted ground gravity values.

A distinctive dataset consists of 1024 GNSS/leveling stations that represent a nice sample of independent data. Ellipsoidal heights of all stations were obtained through GNSS positioning. The accuracy of the estimated geodetic heights can be characterized by the rms error at the order of ±1 cm. Values of the height anomaly are burdened with errors obtained through combination of errors in leveled and GNSS heights and depends namely on quality of the leveled heights. Molodensky normal heights were estimated using very precise leveling from closest points of the Czech national leveling network.

4.2 Results

Point values of the height anomaly obtained through combination of GNSS/leveling heights at the 1024 GNSS/leveling stations are compared against the estimated local quasi-geoid model computed from combination of local ground gravity data and EGM08. The local model of the quasi-geoid is represented by point values of the height anomaly given on the regular grid of geographical coordinates with the equiangular resolution of 30 arcsec. Estimated differences reduced for the mean value due to different values of W_0 used in the definition of EGM08 and the regional vertical datum, are presented in Figure 1. The distribution of the GNSS/leveling stations is also depicted in this figure.

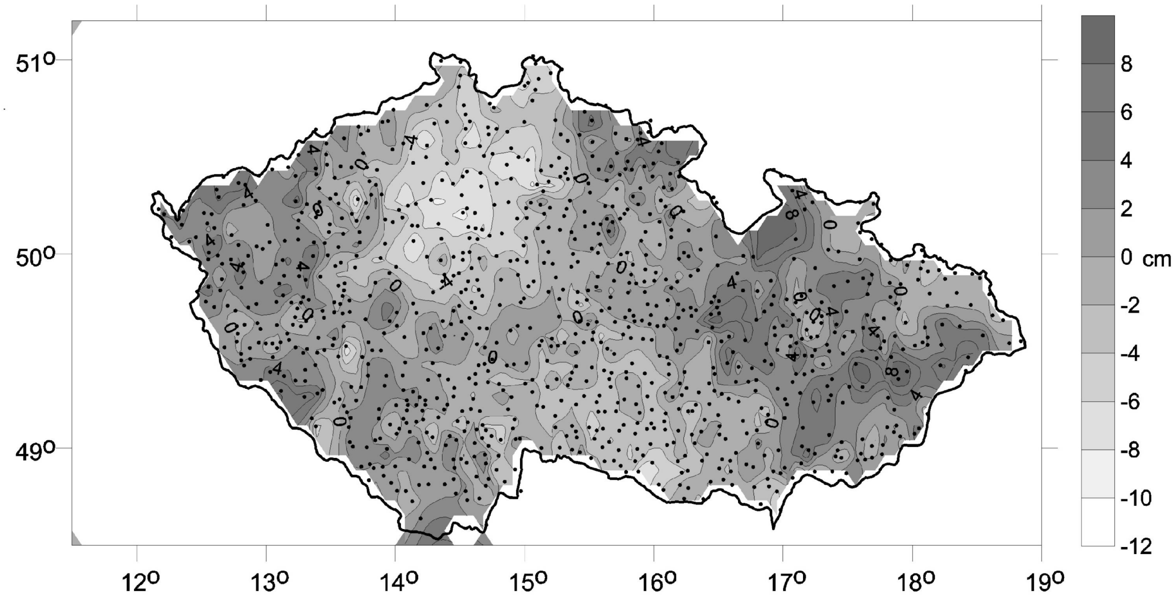

Figure 1: Estimated differences between the gravimetric quasi-geoid and 1024 GNSS/leveling stations (values are in cm).

The results are surprisingly good: the fit of the local quasi-geoid models with the independent GNSS/leveling data is approximately at the level of ±3.5 cm for the Czech part of the test area where the high quality ground gravity data are available. Combining errors related to the GNSS/leveling data and the quasi-geoid model, the fit of the two sets of data seems to be excellent. One can hardly expected a better fit, say at the sub-centimeter level of accuracy, since leveled normal heights, GNSS-based geodetic heights, and estimated height anomalies in the gravimetric quasi-geoid have their respective noise at the centimeter level. In contrary, the comparison of these heights can help to identify possible problems with one of the data source. This is quite important in respect to their intended application for transformation between geometric (geodetic) and physical (normal) heights.

5. CONCLUSIONS

The local quasi-geoid model based on EGM08 and local ground gravity data was compared with respect to 1024 GPS/leveling stations within the area of the Czech Republic. The mean difference and corresponding standard deviation were computed from their differences. The estimated offset can be used for adopting the local quasi-geoid model to the regional height system used over the territory of the Czech Republic (Molodensky normal heights in the realization “Baltic Vertical Datum after Adjustment”). The estimated standard deviation at the level of ±3.5 cm is approximately 4-5 times better than the estimated global accuracy of EGM08 (±15 cm). EGM08 is regionally of much better quality than global estimates suggest, especially over territories with high quality (accuracy and resolution) ground gravity and elevation data (such as the European continent). Looking at the residuals in Figure 1, some local deformations can be seen. They may originate from errors in ground gravity data (local offsets), leveling errors or EGM08. More investigations are needed to explain them adequatly.

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BIOGRAPHICAL NOTES

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